Bases

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February 5, 2016

(Solutions to practice problems are at the end.)

1 Bases

1.1 Definitions

Definition 1 (Base) *A Base defines the counting system used within a problem. By default, a number is in base 10. Base number are expressed as* a_x *where* a *is the number in base* x*.*

Definition 2 (Decimal base) *The decimal base is base 10, the base system we use on Earth when counting.*

Definition 3 (Binary) *The binary base is base 2, the base system that computers use. In binary, only 0s and 1s are used. ie. 010001110*

Definition 4 (Hexadecimal) *The hexadecimal base is base 16, another base system that computers use. In hexadecimal, all 10 digits are used in addition to A,B,C,D,E and F.*

2 What is a base?

The base system defines how we count. Every day, we use the base 10 system to count: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. There are a total of 10 'digits' so we call it base 10. When writing numbers, we can write them as: $A * 10^0 + B * 10^1 + C * 10^2 ... + Z * 10^n$ Every 'digit' in a number increases by a factor of 10. This is what base 10 is.

2.1 Binary

Binary, as mentioned above, is base 2. Instead of using 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, only 0 and 1 are used. A number in base 2 can be expressed as: $A * 2^0 + B * 2^1 + C * 2^2 ... + Z * 2^n$. Every 'digit' in a number increases by a factor of 2. This is what base 2 is.

2.2 Hexadecimal

Hexadecimal, or base 16, is another commonly seen base, since it is a commonly used numbering system in computers. Unlike bases 10 or less, we cannot use 10, 11, etc. to represent the numbers 10-15 in numbers. Instead, we use A, B, C, D, E, and F to represent the numbers 10-15 respectively. ie. $27_{10} = 1B = 11 * 1 + 1 * 16$.

2.3 Converting between bases

The most common application is converting between bases. To convert between bases, we have to go from left to right. Let's convert 53_{10} to base 2 as an example. We know that the powers of 2 are: 1, 2, 4, 8, 16, 32, 64... so we find the largest power that is less than 53, which is 32, and it goes into 53 once so we put a 1 in that place. It can be seen that the number will have 6 digits: $1abcde$. Subtracting 32 from 53, we are left with 21. The next biggest power of 2 that is less than 21 is 16 and it goes into 21 once so we put another one there: $11bcde$. Subtracting 16 from 21, we get 5, which is less than 8, the next power of 2, so we put a 0 there: $110cde$. We still have 5 and 4, the next power of 2, goes into 5 once so we put a 1 next: $1101de$. We are left with 1, which is less than 2 so we put a 0 next: $11010e$. Finally, we put a 1 in the 'digits' place: 110101. So the formal statement will be: $53_{10} = 110101_2$. Using this method, you can convert between any two integer bases very easily.
If you are asked to convert a non-10 base number to another non-10 base number (ie. base 2 to base 5), then you will have to convert the number to base 10 and then to the target base.
Note: in MATHCOUNTS, the bases will usually be less than 10 and if it is greater than 10, it will turn out to be a 'nice' number.

2.3.1 Practice problems

1. Convert 81_{10} to base 5.
2. Convert 432_5 to base 10.
3. Convert 625_7 to base 8.
4. Convert 68_{12} to base 10.
5. Convert 356_{11} to base 7.

3 Solving equations involving bases

Often, especially in MATHCOUNTS, you will be asked to solve equations invoving numbers expressed in differnet bases.

3.1 Given bases, unknown numbers

If you are given the bases of the numbers and asked to find the numerical representation of one of the numbers (ie. $x_5 = 56_8$), The best strategy would be to convert the known numbers to base 10 and then convert to the target base.
Example: $a_4 = b_7 = 1101_2$ Find $a_{10} + b_{10}$.
First, let's solve $1101_2 = x_{10}$. Solving that, we find that $1101_2 = 13_{10}$. Converting that to base 4 and 7, we find that $a = 31_{10}$ and $b = 16_{10}$. Therefore, $a_{10} + b_{10} = 47_{10}$

3.1.1 Practice Problems

1. If some positive integer a is the exact same numbers in the same order in base 6 and base 8, what is the sum of the possible values of a
2. Given $a_3 = b_5 = 34_6$ what is $a_{10} + b_{10}$?
3. What is the decimal difference between 100_{16} and 100_2?

4 Solutions

4.1 2.3.1

1. $81_{10} = 311_5$
2. $432_5 = 117_{10}$
3. $625_7 = 471_8$
4. $68_{12} = 440_{10}$
5. $356_{11} = 1144_7$

4.2 3.1.1

1. For a number to be the exact same numbers in the same order in both base 6 and 8, it has to be a single digit number less than 6. Therefore, the only numbers that meet the criteria are: $1, 2, 3, 4, 5$. Their sum is $\boxed{15}$
2. Given that $a_3 = b_5 = 34_6$ we must first convert 34_6 to base 10. $34_6 = 22_{10}$ Now, converting it to base 3 and 5: $22_{10} = 211_3 = 42_5$ So $a = 211_{10}$ and $b = 42_{10}$. Their sum is $\boxed{253}$
3. Given the two numbers: 100_6 and 100_2, we need the decimal difference so converting them to base 10: $100_6 = 36_{10}$ and $100_2 = 4_{10}$ we see that the decimal difference is: $36 - 4 = \boxed{32}$