Polynomials and Factoring

Jeffrey Shi

January 26, 2016

1 Polynomials

1.1 Definition

Definition 1 (Polynomial) *A polynomial is an algebraic expression involving one or more terms with one or more unknown variables, most commonly labled as x, y, and z.*

1.2 Examples

1. $x + 1$
2. $x^2 - 4$
3. $x^{-1} + x^3 - 5$
4. $x^2 + y^2 + z^2$
5. $\frac{1}{5}y^3 - 4x^2 + xy^2 - 5$

As seen above, polynomials have many different forms; in fact, there are an infinte number of polynomials! The combinations are endless!!

1.3 Monomials

Monomials are polynomials consisting of only 1 term, which will always be a variable.

1.3.1 Examples

1. x
2. $3y$

1.3.2 Solving Monomial equations

When given a Monomial equation such as $5x = 10$, the easiest way to solve it would be to divide both sides by the coefficient of the variable (which is 5 in this case).

1.3.3 Practice Problems

Solve for x, y, or z.

1. $3x = 5$
2. $4z = \frac{4}{5}$
3. $8y = 9\pi$
4. $5xi = \sqrt{-225}$
5. $5^x = 125$

1.4 Binomials

Binomials are polynomials consiting of 2 terms, of which at least 1 must be a variable.

1.4.1 Examples

1. $2x + 5$
2. $3y - 5$
3. $3z - 5x$
4. $iz - z^2$
5. $3xy - 5xyz$

1.4.2 Solving Binomial equations

Solving binomial equations are more difficult than monomial equations. To solve a binomial equation, the variable must be isolated first. ie. $5x - 2 = 13$ In this example, the -2 term on the left must be moved to the right hand side before solving. $5x - 2 = 13 \Rightarrow 5x = 15$ From here, we can solve using the strategy mentioned in 1.3.2.

1.4.3 Practice Problems

1. $2x + 6 = 16$
2. $5z + 9 = 16$
3. $2x + 5 = \sqrt{25} + \sqrt{2}$
4. $2^x + 3 = 11$
5. $2yi + 3i = 11i$

1.4.4 The Binomial Theorem

The Binomial Theorem is a useful theorem that can be used to find specific coefficients within the expansion of a bionomial power.

$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

ie:$(1+2x)^5 = \binom{5}{0}2x^0 + \binom{5}{1}2x^1 + \binom{5}{2}2x^2 + \binom{5}{3}2x^3 + \binom{5}{4}2x^4 + \binom{5}{5}2x^5$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

1.5 Trinomials

Trinomials are the most common polynomials seen in MATHCOUNTS. Trinomials have 3 terms, of which at least 2 are variable terms.

1.5.1 Examples

1. $x^2 + 3x - 4$
2. $3x^2 + 2x - 7$
3. $3x + 2y - 8z$
4. $3xy^2 + 8yz^2 - 4zx^2$
5. $iz^2 - 4z + 2$

1.5.2 Solving Trinomial equations

Solving Trinomial equations can be tricky. There are multiple ways to do it. The first would be to use the quadratic formula: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ if given the trinomial $ax^2 + bx + c$.Note that to use the quadratic forumla, one side muse equal 0. Another way would to be complete the square:

$ax^2 + bx + c \Rightarrow a(x + \frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4}$

ie.$2x^2 + 4x + 8 \Rightarrow 2(x + \frac{4}{2(2)})^2 = -\frac{8}{2} + \frac{4^2}{4} \Rightarrow 2(x+1)^2 = 0$

Note: In MATHCOUNTS, most trinomials will be quadratic equations, a special trinomial in the form of $ax^2 + bx + c$.

1.5.3 Practice Problems

Solve for x, y or z.

1. $x^2 + 2x + 1 = 0$
2. $5x^2 - 10x - 13 = 2$
3. $3z^2 + 7z - 2 = 4$
4. $2^{2y+1} - 2^y - 1 = 0$ (Tip: Substitution)

2 Factoring

Definition 2 (Factoring) *Factoring is taking a polynomial and writing it as the product or sum of 2 or more polynomials.*

2.1 Why is Factoring helpful?

Factoring can be very beneficial in many ways. First of all, it makes it easier to solve a polynomial equation. By setting one side equal to 0 and factoring the other side, the solutions to that particular equation can be easily found. Secondly, factoring is the best way to find the zeroes of a polynomial, most likely a trinomial.

2.2 Factoring Monomials

If a monomial has a coefficient, then to factor it, 'pull out' the coefficient. ie: $5x \Rightarrow 5(x)$.

2.2.1 Practice Problems

Factor into simplest form

1. $6x$
2. iz
3. ey

(Remember that i and e are both constants.)

2.3 Factoring Binomials

Factoring binomials are similar to factoring monomials. To factor a binomial, the two terms must have a gcf greater than 1. If they do not, then the binomial cannot be factored any further. ie. $3x + 5 \Rightarrow 3x + 5$ but $4x + 8 \Rightarrow 4(x + 2)$

2.3.1 Practice Problems

Factor into simplest form

1. $6x + 12$
2. $ix + 5i$
3. $e - e^y$

2.4 Factoring Trinomials

Factoring trinomials can be a bit tricky sometimes. Since there are three terms,it is highly unlikely that there is a common factor between these three terms. Even if there is, it will still be factorable after taking out the common factor.The fail-safe way of factoring a trinomial is to use the quadratic formula, as mentioned above: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ However, in many cases, it would be easier using 'pattern recognition'. Lets take $x^2 + 2x + 1$ as an example. When factoring a trinomial, it will (almost) always factor into $ax^2 + bx + c$. (These will not correspond to $ax^2 + bx + c$. So ultimately, we are trying to find two numbers whose sum is $+2$ and product is $+1$. Obviously, the answer would be 1 and 1 so the factored form would be $(x+1)(x+1)$, which would yield the solution $x = -1$. Of course, it wont always be this easy but it is a useful trick.

2.4.1 Vieta's Formulas

Vietas Formulas is a very unique and useful set of formulas that relate the zeroes to the quadratic function. Vietas formulas state:
If $ax^2 + bx + c = 0$ with zeroes x_1, x_2, then
$x_1 + x_2 = -\frac{b}{c}$
$x_1x_2 = \frac{c}{a}$.
This can be very useful in situations where it would be very difficult to factor the quadratic and when the question is asking for the sum or product of the roots of a quadratic. There is also a form of Vietas for cubic functions:
If $ax^3 + bx^2 + cx + d = 0$ with roots x_1, x_2, x_3,then
$x_1 + x_2 + x_3 = -\frac{b}{a}$
$x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$
$x_1x_2x_3 = \frac{d}{a}$

2.4.2 Practice Problems

Completely Factor

1. $x^2 + 5x - 6$
2. $10x^2 - 23x - 5$
3. What is the product of all real solutions to $x^4 - 1 = 0$?
4. What is the sum of all solutions to $5x^3 - 3x^2 + 10x - 6$

2.5 Special Factorizations

There are a few types of speical polynomials that have a specific rule to factor them.

1. Difference of Squares: $a^2 - b^2 = (a - b)(a + b)$
2. Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

3. Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
4. Sophie Germain Identity: $a^4 + 4b^4 = (a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$

2.5.1 Practice Problems

Completely Factor

1. $4x^2 - 4y^2$
2. $8y^3 - 27z^6$
3. $x^3 + 216z^3$
4. $16x^4 + 324z^4$

3 Solutions

3.1 1.3.3

1. $x = \frac{5}{3}$
2. $z = \frac{1}{5}$
3. $y = \frac{9\pi}{8}$
4. $x = \frac{15i}{5i} = 3$
5. $x = \log_5 125 = 3$

3.2 1.4.3

1. $2x = 10 \Rightarrow x = 5$
2. $5z = 7 \Rightarrow z = \frac{7}{5}$
3. $2x = \sqrt{2} \Rightarrow x = \frac{\sqrt{2}}{2}$
4. $2^x = 8 \Rightarrow x = \log_2 8 = 3$
5. $2yi = 8i \Rightarrow 2y = 8 \Rightarrow y = 4$

3.3 1.5.3

1. $(x+1)^2 = 0 \Rightarrow x = -1$
2. $5x^2 - 10x - 15 = 0 \Rightarrow 5(x-3)(x+1) = 0 \Rightarrow x = \{-1, 3\}$
3. $3z^2 + 7z - 6 = 0 \Rightarrow (3z+2)(z-3) = 0 \Rightarrow z = \{-3, \frac{2}{3}\}$
4. Let $z = 2^y$ then $2z^2 - z - 1 = 0 \Rightarrow (2z+1)(z-1) \Rightarrow z = \{-\frac{1}{2}, 1\}, 2^y = 1, 2^y = -\frac{1}{2} \Rightarrow y = 0$ ($2^y = -\frac{1}{2}$ gives us non-real solutions so it is ignored.

3.4 2.2.2

1. $6x = 6(x)$
2. $iz = i(z)$
3. $ey = e(y)$

(Remember that i and e are both constants)

3.5 2.3.1

1. $6x + 12 = 6(x + 2)$
2. $ix + 5i = i(x + 5)$
3. $e - e^y = e(1 - e^{y-1})$

3.6 2.4.2

1. $(x + 6)(x - 1)$
2. $(2x - 5)(5x + 1)$
3. $x^4 - 1 = (x^2 + 1)(x^2 - 1) \Rightarrow x = \pm 1$ ($x^2 + 1$ is disregarded because it will only yield imaginary solutions)
4. $\frac{3}{5}$

3.7 2.5.1

1. $4(x - y)(x + y)$
2. $(2x - 3z^2)(4x^2 + 6xz^2 + 9z^4)$
3. $(4x^2 + 24x^2z + 9z^2)(4x^2 - 24x^z + 9z^2)$